NASA TECHNICAL TRANSLATION NASA TT F - 11,560

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Translation of (O Lineinykh Differentsial'-
nykh igrakh. II.)
Doklady Akademii Nauk SSSR, Vol. 175, No. 4, pp. 764-766, 1957.

GPO PRICE $ ________

CFSTI PRICE(S) $ ________

Hard copy (HC) 3.00

Microfiche (MF) .65

ff 653 July 65

N68-30718
(ACCESSION NUMBER) (THRU)
5 (PAGES) 1 (CODE)
(NASA CR OR TMX OR AD NUMBER) 19 (CATEGORY)
FACILITY FORM 602

AUG 1968
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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON, D. C. 20546 JULY 1968

NASA TT F-11, 560

LINEAR DIFFERENTIAL GAMES. II.

L. S. Pontryagin
(Akademiya Nauk SSSR, Matematicheskiy Institut, Moscow, USSR).

ABSTRACT: Analysis extending some previous considerations concerning the improvement of results obtained for a certain class of games associated with a vector differential equation dz/dt = Cz + u - v given in a space R of arbitrary dimensionality such that $z \in R$ (C is a square matrix, and u and v are control parameters). The new results are formulated and improved by introducing certain operations involving convex sets and, in particular, the operation of alternate integration of convex sets. All sets examined are closed subsets of the space R.

In the present article, we develop further the ideas expounded in [2] both by broadening the class of games examined and by improving the result. /764*

Let R denote a vector space of arbitrary dimension and let

$$dz/dt = Cz + u - v \tag{1}$$

be a vector differential equation, where $z \in R$. Here, C is a square matrix, and $u \in P$ and $v \in Q$ are control parameters, where P and Q are closed bounded convex subsets, of arbitrary dimension, of the space R. Further, let M denote a closed convex subset, again of arbitrary dimension, of the space R. Equation (1) and the set M define a differential game. Here, u is the pursuing parameter, v is the fleeing parameter, and M is the set on which the game is terminated [1].

To formulate and prove the result, we introduce certain operations on convex sets, in particular, the operation of alternate integration of convex sets. All the sets that we shall consider in what follows are closed subsets of the space R.

A. Let A and B denote two sets and let α and β denote two real numbers. We denote by

$$\alpha A + \beta B \tag{2}$$

the set of all vectors $\alpha x + \beta y$, where $x \in A$ and $y \in B$. In the case $\alpha = \beta = 1$, formula (2) gives the algebraic sum of the sets. In the case $\alpha = 1$, $\beta = -1$,

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formula (2) gives the algebraic difference of the sets. If A and B are convex sets, formula (2) yields convex sets.

We denote by

$$A \overset{*}{-} B \tag{3}$$

the set of all vectors x satisfying the condition $x + B \subset A$. The set (3) may happen to be empty. If A is a convex set, formula (3) also defines a convex set. Suppose that A, U, and V are convex sets. Then, as one can easily show,

$$(A \overset{*}{-} U) \overset{*}{-} V = A \overset{*}{-} (U + V), \tag{4}$$

$$(A + U) \overset{*}{-} V \supset (A \overset{*}{-} V) + U. \tag{5}$$

B. Let A_0 denote a convex set and let

$$U_1, \ldots, U_n; \quad V_1, \ldots, V_n \tag{6}$$

denote two sequences of convex sets. We define inductively the set A_{i+1}, for $i = 0, 1, \ldots, n-1$, by

$$A_{i+1} = (A_i + U_{i+1}) \overset{*}{-} V_{i+1}. \tag{7}$$

It is natural to call the set A_n the alternate sum of the sequences (6) with initial value A_0. Define

$$U = U_1 + \ldots + U_n; \quad V = V_1 + \ldots + V_n,$$

Then, it follows from formulas (4) and (5) that /765

$$A_n \subset (A_0 + U) \overset{*}{-} V. \tag{8}$$

C. Suppose that $A = A_0$ is a convex set and that $U(\tau)$ and $V(\tau)$ are two bounded convex sets that depend continuously on a real parameter τ on the interval $p \leqslant \tau \leqslant q$. Let us define the alternate integral of the functions $U(\tau)$ and $V(\tau)$

$$B = \int_{A,\,p}^{q} [U(\tau)\,d\tau \overset{*}{-} V(\tau)\,d\tau]. \tag{9}$$

Here, A is the initial set of integration, p is the initial value of τ, and q is its final value. The integral (9) itself is a convex set. To define the integral (9), we partition the interval of integration into small subintervals by means of points $r_0 = p, r_1, \ldots, r_n = q$. Define

$$U_i = \int_{r_{i-1}}^{r_i} U(\tau)\,d\tau; \quad V_i = \int_{r_{i-1}}^{r_i} V(\tau)\,d\tau \quad (i = 1, \ldots, n). \tag{10}$$

These integrals of convex sets are defined in a natural manner on the basis of the operation of addition (2).

Beginning with the sequences (10) and the initial set A_0, let us construct the alternate sum A_n (see part B). The limit of this alternate sum as the length of the interval $p \leqslant \tau \leqslant q$ approaches zero is the integral (9). Now, suppose that the functions $U(\tau)$ and $V(\tau)$ are defined on the interval $p \leqslant \tau \leqslant r$, where $r > q$. Then, we have the inclusion relationship

$$\int_{A,\,p}^{r} [U(\tau)\,d\tau \stackrel{*}{-} V(\tau)\,d\tau] \subset \left(B + \int_{q}^{r} U(\tau)\,d\tau\right) \stackrel{*}{-} \int_{q}^{r} V(\tau)\,d\tau \tag{11}$$

[cf. (8) and (9)].

Theorem. Let us set A = -M [cf. (1)] and let us set up the alternate integral

$$W(\tau) = \int_{A,\,0}^{\tau} [e^{rC} P\,dr \stackrel{*}{-} e^{rC} Q\,dr] \quad (\tau > 0). \tag{12}$$

Let z_0 denote an arbitrary point of the space R not in M. Define $\eta(\tau) = e^{\tau C} z_0$. If, for some $\tau > 0$, the point $-\eta(\tau)$ belongs to a convex set $W(\tau)$, let us denote by τ_0 the smallest value of τ for which this is true. Then the game (1), which begins at the point z_0 can be terminated after a period of time not exceeding the number

$$T(z_0) = \tau_0. \tag{13}$$

In the proof of this theorem, the control u will be constructed with consideration of the control v in order to shorten the game time as much as possible. In the construction of the control u(t) at the instant t, we shall use the value of z(t) at the same instant and the control v(s) on the interval $t \leqslant s \leqslant t + \varepsilon$, where ϵ is an arbitrarily small positive number (cf. [2]).

Proof: We shall assume that the control v(t) is defined on the interval $0 \leqslant t \leqslant \varepsilon$. Let u(t) denote a control (for the moment, arbitrary) defined on the same interval. When we substitute these equations into equation (1), let us find its solution z(t) on the interval $0 \leqslant t \leqslant \varepsilon$ subject to the initial condition $z(0) = z_0$. The number $T(z(\varepsilon))$ (cf. (13)) is a functional of the function u(t). Below, we shall choose the function u(t) in such a way that the number $T(z(\varepsilon))$ is minimized and we shall show that /766

$$T(z_0) - T(z(\varepsilon)) \geqslant \varepsilon. \tag{14}$$

It follows from (11) that, for arbitrary $\tau \geqslant \varepsilon$, we have

$$\begin{aligned} W(\tau) &\subset \left(W(\tau - \varepsilon) + \int_{\tau-\varepsilon}^{\tau} e^{rC} P\,dr\right) \stackrel{*}{-} \int_{\tau-\varepsilon}^{\tau} e^{rC} Q\,dr \subset \\ &\subset W(\tau - \varepsilon) + \int_{\tau-\varepsilon}^{\tau} e^{rC} P\,dr - \int_{\tau-\varepsilon}^{\tau} e^{rC} v(\tau - r)\,dr = D(\tau). \end{aligned} \tag{15}$$

It should be noted that the last term of formula (15) is defined for all $\tau > \varepsilon$, since the function $v(\tau - r)$ is defined on the entire interval of integration $\tau - \varepsilon \leqslant r \leqslant \tau$, because its argument on this interval of integration varies over the interval $[0, \varepsilon]$ and the function v(t) is defined on that interval. By our assumption, the point $-\eta(\tau)$ belongs to the left-hand member of the inclusion relationship (15) when $\tau = \tau_0$. Let $\tau_1 \leqslant \tau_0$ denote the smallest value of τ for which the point $-\eta(\tau)$ belongs to the last member of (15), namely the set $D(\tau)$. Then, there exists a function $u(t) \in P$, for $0 \leqslant t \leqslant \varepsilon$, such that the point $-e^{(\tau_1-\varepsilon)C}z_1$ belongs to the set $W(\tau_1 - \varepsilon)$ under the condition that

$$z_1 = e^{\varepsilon C} z_0 + \int_0^{\varepsilon} e^{sC} (u(\varepsilon - s) - v(\varepsilon - s))\, ds. \tag{16}$$

Since obviously $z_1 = z(\varepsilon)$, this proves the assertion.

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Translated for the National Aeronautics and Space Administration by Scripta Technica, Inc., NASw-1694.